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# Two Generalizations of Posets of Shuffles[1]

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We study posets defined by Stanley as a multiset generalization of Greene's posets of shuffles. Ehrenborg defined a quasi-symmetric function encoding for the flag $f$-vector, denoted $F_P$, and we determine $F_P$ for shuffle posets of multisets,



shuffles to posets for shuffling $k$ words, answering a question of Stanley. Finally, we extend our results about shuffle posets of multisets to $k$-shuffle posets. © 2001 Elsevier Science

## 1. INTRODUCTION

Greene introduced posets of shuffles in [Gr, pp. 191–192], and we study a multiset generalization suggested by Stanley [St4]. Let us first review Greene's definition for posets of shuffles. Let $A_1 = \{a_1, ..., a_m\}$ and $A_2 = \{b_1, ..., b_n\}$ be two disjoint alphabets, let $w_1$ be the word $a_1 a_2 \cdots a_m$ and let $w_2$ be the word $b_1 b_2 \cdots b_n$. We obtain shuffled words by interspersing the letters of $w_1$ with the letters of $w_2$ and denote such a shuffled word by $w_1 \sqcup\!\sqcup\, w_2$. Thus, each possible shuffled word $w_1 \sqcup\!\sqcup\, w_2$ when restricted to $A_1$ and $A_2$, respectively, must satisfy $w_1 \sqcup\!\sqcup\, w_2|_{A_1} = w_1$ and $w_1 \sqcup\!\sqcup\, w_2|_{A_2} = w_2$.

The structure of the shuffle poset given by the words $w_1$ and $w_2$ depends only on the lengths of $w_1$ and $w_2$, so let us denote a traditional shuffle poset by $W_{1^m, 1^n}$. The elements of $W_{1^m, 1^n}$ are the set of subwords of all possible shuffled words $w_1 \sqcup\!\sqcup\, w_2$. We denote these subwords by $u_1 \sqcup\!\sqcup\, u_2$ where $u_1 \sqcup\!\sqcup\, u_2|_{A_1} = u_1$ is a subword of $w_1$ and $u_1 \sqcup\!\sqcup\, u_2|_{A_2} = u_2$ is a subword of $w_2$. Let $\hat{0} = w_1$, $\hat{1} = w_2$ and let there be a covering relation $u \prec v$ whenver $v$ is obtained from $u$ by either deleting a letter belonging to $A_1$ or

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inserting a letter belonging to $A_2$ in a way that produces a poset element. It is implicit to this definition that each letter occurs with multiplicity one.

Stanley [St4] generalized this definition by allowing repetition of letters in the words to be shuffled, keeping the constraint that the words $w_1$ and $w_2$ to be shuffled come from disjoint alphabets; he also imposed the requirement that identical letters must always occur consecutively in shuffled words. This requirement leads to posets with a great deal more structure than shuffling words arbitrarily would yield. Let the composition $\alpha=(\alpha_1, ..., \alpha_k)$ be the **type** of the word $w=a_1^{\alpha_1} \cdots a_k^{\alpha_k}$. Suppose two words $w_1$ and $w_2$ from disjoint alphabets are of type $\alpha$ and $\beta$, respectively. These two compositions will determine the shuffle poset of multisets given by $w_1$ and $w_2$ up to isomorphism, so we denote this poset by $W_{\alpha, \beta}$. As in traditional shuffle posets, a shuffled word $w$ is an element of $W_{\alpha, \beta}$ if $w$ restricted to the alphabet $A_1$ is a subword of $w_1$ and $w$ restricted to the alphabest $A_2$ is a subword of $w_2$, but with the additional requirement that identical letters must occur consecutively.

For example, if $w_1=aaab$ and $w_2=c$, which means $\alpha=(3, 1)$ and $\beta=(1)$, then $aacb$ is a valid poset element while $acab$, $ba$ and $aaaa$ are not. Similarly to traditional shuffle posets, $w_1$ is the minimal element, $w_2$ is the maximal element, and there is a covering relation $u \prec v$ whenever $v$ is obtained from $u$ by either deleting a letter of $w_1$ or legally inserting a letter of $w_2$. Figure 1 illustrates the poset $W_{(3), (2)}$ with $w_1=aaa$ and $w_2=bb$. The traditional shuffle posets are usually denoted $W_{m, n}$, but unfortunately in the notation of shuffle posets of multisets this necessarily becomes $W_{1^m, 1^n}$. We note that a different generalization of shuffle posets based on a shuffling operation for lattices has been examined by Doran in [Do].

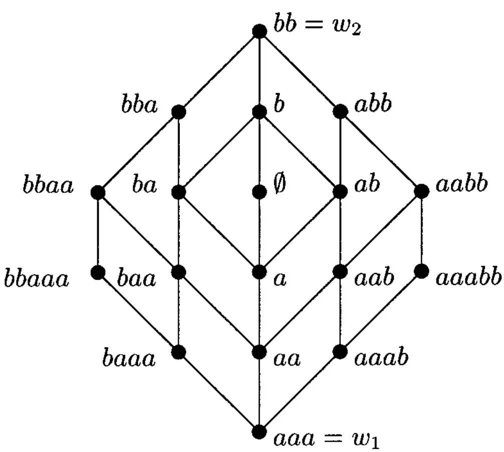


**FIG. 1.** The shuffle poset of multisets $W_{3, 2}$.

Greene's definition was motivated by a very idealized model of DNA mutation. Posets of shuffles are designed to reflect the space of all minimal paths mutating a word $\alpha$ to a word $\beta$ by inserting and deleting letters. While shuffle posets of multisets allow repetition of letters, very little is known about posets of shuffled words in which identical letters need not be consecutive or posets in which $w_1$ and $w_2$ are allowed to have letters in common; it seems likely that such posets would lack most of the nice structural features with which shuffle posets of multisets are endowed.

Our plan is to give properties of shuffle posets of multisets based on a topological decomposition of the order complex. In particular, we provide a flag $f$-vector formula, other combinatorial formulas, an EL-labelling, a symmetric chain decomposition, and a description of $M$-chains, thus proving supersolvability. We also verify that the shuffle posets of multisets are lattices. In Sections 4 and 5, we answer a question of Stanley by generalizing posets of shuffles to posets for shuffling $k$ words. We verify that $k$-shuffle posets may be defined consistently and then extend all of our results for shuffle posets of multisets to $k$-shuffle posets. Our flag $f$-vector formula derivation will lead to most of our results, so we begin with background terminology followed by a description of the encoding for the flag $f$-vector which we shall use, after some brief background terminology.

The order complex of a poset is the simplicial complex comprised of an $(i-1)$-face for each $i$-chain $\hat{0} < v_1 < \cdots < v_i < \hat{1}$; it is pure of dimension $d$ if each maximal face has dimension $d$. A total order $F_1, \ldots, F_m$ on the facets of a pure simplicial complex is a shelling if each intersection $F_j \cap (\bigcup_{1 \leqslant i \leqslant j-1} F_i)$ is itself pure of codimension one. One way to prove that the order complex of a poset is shellable is to provide an EL-labelling, defined as follows:

Definition 1.1. An edge-labelling $\lambda$ of a finite, graded poset is an **EL-labelling** if it satisfies the following two properties.

(1) For each $u \leqslant v$ there is a unique saturated chain $u = u_1 \prec \cdots \prec u_k = v$ such that $\lambda(u_1, u_2) \leqslant \cdots \leqslant \lambda(u_{k-1}, u_k)$.

(2) Given any other saturated chain from $u$ to $v$, the word given by its sequence of edge labels is lexicographically larger than the word $\lambda(u_1, u_2) \cdots \lambda(u_{k-1}, u_k)$.

A simplicial complex $\Delta$ is **Cohen–Macaulay** if the link of each face $F$ has reduced homology groups $\tilde{H}_i(\mathrm{lk}(F)) = 0$ for $i < dim(\mathrm{lk}(F))$. A poset is **Cohen–Macaulay** if its order complex is Cohen–Macaulay.

A lattice is **supersolvable** if it has a chain known as an **M-chain**, namely a saturated chain $C$ such that the sublattice generated by $C$ and other other chain in $L$ is distributive. Supersolvability implies EL-shellability which in turn implies Cohen–Macaulayness.

We shall prove these properties for generalized shuffle posets. Our starting point is to express the flag $f$-vector as a quasi-symmetric function. A power series $q(x)$ is **quasi-symmetric** if the coefficient of $x_{a_1}^{k_1} \cdots x_{a_n}^{k_n}$ in $q(x)$ equals the coefficient of $x_{b_1}^{k_1} \cdots x_{b_n}^{k_n}$ for any $a_1 < \cdots < a_n$ and any $b_1 < \cdots < b_n$, together with any choice of $k_1, ..., k_n \in \mathbb{N}$.

Recall the quasi-symmetric function encoding

$$F_P = \sum_{\hat{0} = t_0 \leqslant t_1 \leqslant \cdots \leqslant t_{k-1} < t_k = \hat{1}} x_1^{\rho(t_0, t_1)} x_2^{\rho(t_1, t_2)} \cdots x_k^{\rho(t_{k-1}, t_k)}$$

for the flag $f$-vector of a finite, nontrivial ranked poset with $\hat{0}$ and $\hat{1}$, as introduced by Ehrenborg in [Eh, pp. 9–10]. In this expression, $\rho(x, y)$ denotes the difference in the ranks of $x$ and $y$, and the sum is over all multichains of any length, as long as they include at least one copy of $\hat{0}$ and exactly one copy of $\hat{1}$. Stanley noted that $F_P$ is a symmetric function whenever $P$ is locally rank-symmetric in [St4, pp. 4–5]. Ehrenborg observed in [Eh, p. 10] that $F_{P \times Q} = F_P F_Q$. Ehrenborg also noticed that $F_P = h_n$ for $P$ a chain of rank $n$, since each possible monomial of degree $n$ is given by a single multichain in $C_{n+1}$. Combining these facts shows that $F_P = h_\lambda$ when $P$ is the product of chains $C_{\lambda_1+1} \times \cdots \times C_{\lambda_k+1}$.

To determine $F_P$ for generalized shuffle posets, we will use these observations along with an interpretation for the skew-Schur functions as regions in the order complex of the boolean lattice $B_n$. The elements of $B_n$ naturally correspond to the subsets of a set $\{a_1, ..., a_n\}$, and each multichain $\hat{0} = v_0 \leqslant v_1 \leqslant \cdots \leqslant v_{k-1} < v_k = \hat{1}$ thereby gives rise to a string of inclusions $\emptyset = S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{k-1} \subset S_k = \{a_1, ..., a_n\}$. Let $V$ be a real vector space with coordinates $a_1, ..., a_n$. We assign each multichain in $B_n$ to an intersection of hyperplanes and open half-spaces restricted to the hyperplane $\sum_{i=1}^n a_i = 0$ in $V$ as follows. Partition $\{a_1, ..., a_n\}$ into blocks $B_1, ..., B_k$ by letting $B_i = S_i \backslash S_{i-1}$ for $1 \leqslant i \leqslant k$. A multichain is then assigned to the intersection of all the hyperplanes $a_j = a_{j'}$ such that $a_j, a_{j'} \in B_i$ for $1 \leqslant i \leqslant k$ with all the open half-spaces $a_j < a_{j'}$ such that $a_j \in B_i$ and $a_{j'} \in B_{i'}$ for $1 \leqslant i < i' \leqslant k$.

The hyperplane arrangement given by the hyperplanes $a_i = a_j$ for $1 \leqslant i < j \leqslant n$ decomposes $V$ into cones which are bounded by these hyperplanes. Let us restrict this decomposition of $V$ to the $(n-2)$-sphere which is obtained by taking the slice of the unit sphere $\sum_{i=1}^n a_i^2 = 1$ which intersects the hyperplane $\sum_{i=1}^n a_i = 0$. The hyperplane arrangement thereby specify a triangulation of the $(n-2)$-sphere which by definition consists of the same simplices as the order complex for $B_n$. Each $i$-chain in $B_n$ gives rise to an $(i-1)$-face in its order complex, and our assignment sends each chain to a region of dimension $i-1$ which indeed corresponds to the $(i-1)$-face of the order complex. This discussion is informed by a similar point of view in [HRW, pp. 5–11].

We may similarly view elements of $B_n$ as the subsets of a set $\{a_{i,j} \mid j \leqslant \lambda_i\}$ given by any partition $\lambda \vdash n$. This allows us to define a system of inequalities based on the fact that the entries in semi-standard Young tableaux (SSYT) of shape $\lambda/\mu$ must increase weakly in rows and strictly in columns. Lemma 1.1 will show that the chains in a region specified by the following such system of constraints contributes to $F_P$ monomials whose sum is $s_{\lambda/\mu}$. Let $S = \{a_{i,j} \mid \mu_i < j \leqslant \lambda_i\}$; whenever the Young diagram of shape $\lambda/\mu$ includes a pair of neighboring boxes in positions $(i, j)$ and $(i+1, j)$, we introduce a weak inequality $a_{i,j} \leqslant a_{i+1,j}$, and for each pair of neighboring boxes in positions $(i, j)$ and $(i, j+1)$, we establish a strict inequality $a_{i,j} < a_{i+1,j}$. Let $n = l - k$ for $\mu \subseteq \lambda$ satisfying $\mu \vdash k$ and $\lambda \vdash l$.

Lemma 1.2. *The sum $F_{B_n}$ restricted to multichains satisfying the constraints described above for the skew-shape $\lambda/\mu$ is equal to the skew-Schur function $s_{\lambda/\mu}$.*

*Proof.* This is a direct consequence of the combinatorial definition of skew-Schur function. Considering $s_{\lambda/\mu}$ as a sum over the semi-standard Young tableaux of shape $\lambda/\mu$, we need only show that each such tableau giving rise to a monomial of content $v$ corresponds to a multichain in the bounded region which contributes $x^v$ to $F_P$, and that this correspondence is a bijection. The bijection comes from placing the number $d$ in the box at position $(i, j)$ in a SSYT of shape $\lambda/\mu$ for $a_{i,j} \in S_d \backslash S_{d-1}$ in the corresponding multichain $\emptyset = S_0 \subseteq \cdots \subseteq S_{k-1} \subset S_k = \{a_{i,j} \mid \mu_i < j \leqslant \lambda_i\}$. The constraints on multichains in a region are designed to correspond to the constraints on semi-standard Young tableaux entries so that legal multichains are mapped to legal SSYT. The monomials will agree because $|S_d| - |S_{d-1}|$ for a multichain will be the number of boxes containing $d$ in the corresponding SSYT. ∎

One reason to be interested in when $F_P$ is a Schur-positive symmetric function is the following observation of Stanley: whenever $F_P$ is a symmetric function, the number of maximal chains in $P$ equals the dimension of the virtual symmetric group representation with $F_P$ as Frobenius characteristic, so there could be a symmetric group action permuting maximal chains which has Frobenius characteristic equalling $F_P$ or $\omega F_P$ where $\omega$ denotes the symmetric function involution which sends each Schur function $s_\lambda$ to the Schur function $s_{\lambda^T}$ of transpose shape. Such symmetric group actions are discussed in more detail in [SS].

Lemma 1.1 and related topological interpretations for the symmetric function bases are discussed more thoroughly in [He].

## 2. A DECOMPOSITION AND CONSEQUENT FLAG *f*-VECTOR FORMULA

We begin with a simple, but hopefully suggestive example.

Let $w_1 = aaa$ and $w_2 = bb$, as in Fig. 2. We decompose the space of maximal chains into two pieces and account for the contribution of each separately to $F_P$. Chains in which each element is a subword of $w_1 = aaa$ followed by a subword of $w_2 = bb$ are exactly the chains in the product of a 3-chain with a 4-chain, shown on the left in Fig. 2. Hence, these contribute $h_3 h_2 = s_{\square\square\square} s_{\square\square}$ to $F_P$.

The chains with the letter $b$ occurring immediately before the letter $a$ in some element of the chain give another copy of $h_3 h_2$ for the product of chains from subwords of $bb$ followed by subwords of $aaa$, shown on the right in Figure 2. We must subtract for overlap, which means chains in which $a$ and $b$ never appear together. These are the chains contained in the maximal chain $aaa < aa < a < \emptyset < b < bb$, so we subtract $h_5$ to obtain $F_P = h_3 h_2 + (h_3 h_2 - h_{3+2}) = s_{\square\square\square} s_{\square\square} + s_{\begin{smallmatrix} & & \square & \square \\ \square & \square & \square & \end{smallmatrix}}$ for $P = W_{3,2}$.

We may embed either piece of the decomposition into a boolean lattice with atoms $a_1, a_2, a_3, b_1, b_2$ by imposing the constraints $a_1 \leqslant a_2 \leqslant a_3$ and $b_1 \leqslant b_2$. For the latter piece of the decomposition, we also need the strict inequality $b_1 < a_3$ to represent the requirement that not all three copies of $a$ be deleted before the first copy of $b$ is inserted. Thus, we require the skew-shape entries

$$\begin{array}{cccc} & & b_1 & b_2 \\ a_1 & a_2 & a_3 & \end{array}$$

to be weakly increasing in rows and strictly increasing in columns.

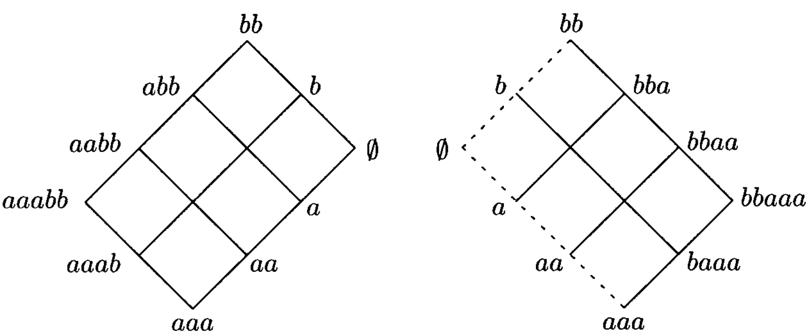


**FIG. 2.** A chain decomposition for $W_{3,2}$.

Let $m$ and $n$ be the lengths of $\alpha$ and $\beta$, respectively, and let $\gamma$ be the composition obtained by concatenating the compositions $\alpha$ and $\beta$. We shall use the weak order (cf. [Hu] for definition and properties) interval from $(n, ..., 1, m+n, ..., n+1)$ to the reverse permutation $(m+n, ..., 1)$ to order the set of words which may be obtained by shuffling a word $w_1$ of type $\alpha$ with a word $w_2$ of type $\beta$; note that permutations in this interval correspond to shuffled words by sending $n, ..., 1$ to the letters of $w_2$ and $m+n, ..., n+1$ to the letters of $w_1$ since the interval preserves the relative order of the first $n$ letters and of the last $m$ letters.

Our chain decomposition for $W_{\alpha, \beta}$ has four steps:

(1) Break a shuffle poset of multisets into overlapping products of chains. Each shuffled word $w = w_1 \sqcup\!\sqcup w_2$ gives rise to such a subposet $P_w$ consisting of all the subwords of $w_1 \sqcup\!\sqcup w_2$. The poset $P_w$ is a product of chains $C_\gamma$.

(2) Partially order the $P_w$ by partially ordering the shuffled words specifying them. Using one-line notation, our partial order on shuffled words is the interval in the weak order from $(n, ..., 1, m+n, ..., n+1)$ to the reverse permutation $(m+n, ..., 1)$. (This goes against the usual convention of swapping adjacent values in weak order covering relations; in studying shuffled words, it seems more natural to swap adjacent positions which amounts to taking the weak order interval on inverse permutations.)

(3) Assign each poset chain $C$ to the earliest product of chains containing it, i.e. to the meet in the weak order of all the products of chains $P_w$ containing the chain. Equivalently, this will turn out to be the $P_w$ with exactly the set of interface pairs of the chain $C$.

(4) Compute $F_P$ for each piece of the decomposition and sum the results.

This decomposition leads to the following formula for $F_P$ in terms of skew-Schur functions.

THEOREM 2.1. *The flag f-vector $F_P$ for $P = W_{\alpha, \beta}$ is*

$$\sum_{j=0}^{\min(l(\alpha), l(\beta))} \sum_{\substack{1 \leqslant a_1 < \cdots < a_j \leqslant l(\alpha) \\ 1 \leqslant b_1 < \cdots < b_j \leqslant l(\beta)}} \left( \prod_{i=1}^{j} s_{(\alpha_{a_i} + \beta_{b_i} - 1, \beta_{b_i})/(\beta_{b_i} - 1)} \right)$$

$$\times \left( \prod_{i \notin \{a_1, ..., a_j\}} s_{\alpha_i} \right) \left( \prod_{i \notin \{b_1, ..., b_j\}} s_{\beta_i} \right).$$

*Equivalently,* $F_P$ *may be expressed in terms of complete symmetric functions as*

$$\sum_{j=0}^{\min(l(\alpha),\, l(\beta))} \sum_{\substack{1 \leqslant a_1 < \cdots < a_j \leqslant l(\alpha) \\ 1 \leqslant b_1 < \cdots < b_j \leqslant l(\beta)}} \left( \prod_{i=1}^{j} h_{\alpha_{a_i}} h_{\beta_{b_i}} - h_{\alpha_{a_i} + \beta_{b_i}} \right) \times \left( \prod_{\substack{i \notin \{a_1, ..., a_j\} \\ 1 \leqslant i \leqslant l(\alpha)}} h_{\alpha_i} \right) \left( \prod_{\substack{i \notin \{b_1, ..., b_j\} \\ 1 \leqslant i \leqslant l(\beta)}} h_{\beta_i} \right).$$

In the special case of traditional shuffle posets, namely $W_{1^m, 1^n}$, this becomes

$$F_P = \sum_{j=0}^{\min(m,n)} \binom{m}{j} \binom{n}{j} e_2^j \, e_1^{n+m-2j},$$

so we recover a formula of [SS, p. 21] for traditional shuffle posets. Simply substitute $e_2$ for $h_1 h_1 - h_2$ and $e_1$ for $h_1$ above.

Let us extend Greene's notion of the interface of a poset element in [Gr, pp. 195–196] to chains, in which context poset elements are 1-chains. Recall that the **interface** of a traditional shuffle poset element $u_1 \sqcup\!\sqcup\, u_2$ is the collection of pairs of letters $(a, b)$ such that $a$ belongs to $w_2$, $b$ belongs to $w_1$, and $a$ immediately precedes $b$ in the shuffled word $u_1 \sqcup\!\sqcup\, u_2$, so the interface determines the degree to which $u_1$ and $u_2$ are shuffled. Although letters may occur with multiplicity in shuffle posets of multisets, we refer to pairs of letters as belonging to the interface when we actually mean pairs of classes of identical letters, since identical letters always occur consecutively.

Definition 2.2. The **interface** of a chain is obtained by taking the union of the interfaces of all chain elements, then eliminating those ordered pairs which are preempted by other "more shuffled" pairs arising elsewhere in the chain. One pair preempts another if it consists of the same letter or a later letter of $w_2$ and the same letter or an earlier letter of $w_1$. The interface does not depend on the order in which pairs are eliminated because this notion of preemption is transitive.

Following [Gr, pp. 195–196], letters not occurring in the interface of a chain comprise the **residue** of the chain. As we mentioned before, each possible shuffled word $w_1 \sqcup\!\sqcup\, w_2$ yields a product of chains sublattice consisting of all subwords of this shuffled word. Partitioning chains according to their interface amounts to assigning each chain to the product of chains containing it which comes earliest in this partial order, namely the one specified by the least shuffled word.

*Proof of Theorem* 2.1. Each summand in Formula 1 accounts for chains with a particular interface, specified by sets $\{a_1, ..., a_j\} \subseteq \{1, ..., l(\alpha)\}$ and $\{b_1, ..., b_j\} \subseteq \{1, ..., l(\beta)\}$ which index the distinct letters from $w_1$ and $w_2$, respectively. The collection of multichains with this particular interface will be the chains in the product of chains $P_w$ where $w$ has exactly this interface and the chains satisfy the following constraints. When a letter $r_i$ occurs with multiplicity $k$, we denote the $k$ copies by $r_{i_1}, ..., r_{i_k}$ and impose the constraints $r_{i_1} \leqslant \cdots \leqslant r_{i_k}$ on multichains, so as to embed $P_w$ in a boolean lattice. Furthermore, for each interface pair $(a_i, b_i)$ of letters occurring with multiplicities $m$ and $n$, respectively, there is a constraint $b_{i_1} < a_{i_m}$ to reflect the fact that the first copy of $b_i$ must be inserted before the last copy of $a_i$ is deleted; otherwise, the multichain be consistent with some earlier $w'$ not containing this interface pair. Recall that $s(m+n-1, n)/(m-1) = h_m h_n - h_{m+n}$, so this gives the expression in terms of complete symmetric functions. ∎

EXAMPLE. *Let $w_1 = 12222$ and $w_2 = aabbb$. On the left side in Fig.* 3, *we partially order the shuffled words specifying the product of chain sublattices. On the right, we record their corresponding contributions to $F_P$, so in this case $F_P$ is the sum of these complete symmetric functions.*

COROLLARY 2.1. *$F_P$ is Schur-positive for shuffle posets of multisets.*

*Proof.* Skew-Schur functions are Schur-positive, and the Littlewood–Richardson Rule implies that products of Schur functions are also Schur-positive. ∎

Denote by (1) the formula for $F_P$ for shuffle posets of multisets

$$\sum_{j=0}^{\min(l(\alpha), l(\beta))} \sum_{\substack{1 \leqslant a_1 < \cdots < a_j \leqslant l(\alpha) \\ 1 \leqslant b_1 < \cdots < b_j \leqslant l(\beta)}} \left( \prod_{i=1}^{j} h_{\alpha_{a_i}} h_{\beta_{b_i}} - h_{\alpha_{a_i} + \beta_{b_i}} \right) \times \left( \prod_{\substack{i \notin \{a_1, ..., a_j\} \\ 1 \leqslant i \leqslant l(\alpha)}} h_{\alpha_i} \right) \left( \prod_{\substack{i \notin \{b_1, ..., b_j\} \\ 1 \leqslant i \leqslant l(\beta)}} h_{\beta_i} \right). \tag{1}$$

Recall from [Gr, p. 200] that a chain is $w_1$-*terminal* (resp. $w_2$-*terminal*) if each chain element involving the last letter of $w_1$ (resp. $w_2$) has this letter occurring last, i.e., after all letters of $w_2$ ($w_1$) appearing in the shuffled word. This leads to the following recursive formula, in which $F_{\alpha, \beta}$ denotes $F_{W\alpha, \beta}$.

LEMMA 2.1. *The shuffle posets of multisets satisfy the recurrence*

$$F_{\alpha, \beta} = F_{\alpha - \alpha_k, \beta}\, h_{\alpha_k} + F_{\alpha, \beta - \beta_l}\, h_{\beta_l} - F_{\alpha - \alpha_k, \beta - \beta_l}\, h_{\alpha_k + \beta_l} \tag{2}$$

*for $\alpha = (\alpha_1, ..., \alpha_k)$ and $\beta = (\beta_1, ..., \beta_l)$.*

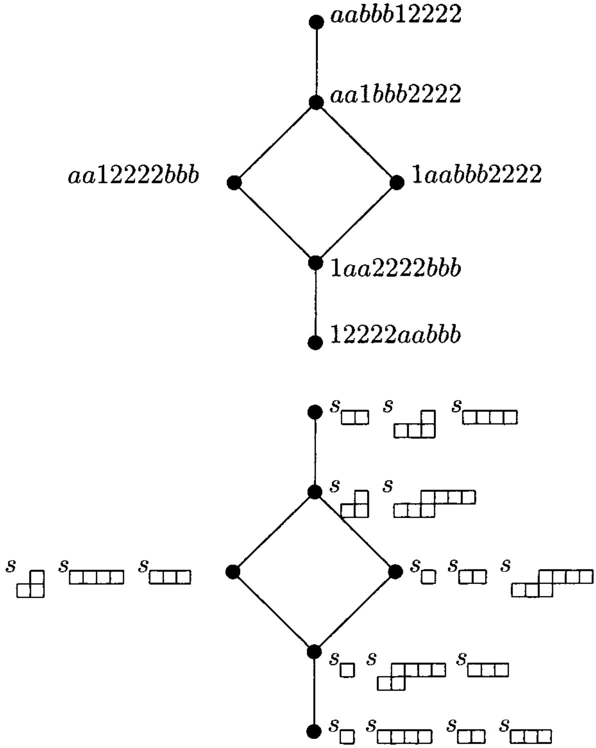


**FIG. 3.** Summing contribution to $F_P$.

This yields following generating function.

THEOREM 2.2. *Summing over compositions* $\alpha$ *and* $\beta$ *indexed by monomials in noncommutative variables* $u_i$ *and* $v_j$ *with relations* $u_i v_j = v_j u_i$ *for* $i, j > 0$ *yields*

$$\sum_{\alpha, \beta} F_{\alpha, \beta} u_{\alpha_1} \cdots u_{\alpha_k} v_{\beta_1} \cdots v_{\beta_l} = \frac{1}{1 - \sum_{i > 0} (u_i + v_i)\, h_i + \sum_{i, j > 0} u_i v_j h_{i+j}}.$$

Non-commutative variables are used because $F_P$ depends on the order of the parts of the compositions $\alpha$ and $\beta$. This agrees with a result of [SS, p. 9] when we express $h_1$ as $e_1$, $h_2$ as $e_1 e_1 - e_2$, let $u_1 = u$, $v_1 = v$ and set $u_i = v_i = 0$ for $i > 1$.

The symmetry of the right hand side in Theorem 2.2 in the alphabets $u$ and $v$ immediately implies that $F_{\alpha, \beta} = F_{\beta, \alpha}$. This is also a consequence of local rank-symmetry, but the analogous result for $k$-shuffle posets will not follow from local rank-symmetry.

## 3. PROPERTIES OF SHUFFLE POSETS OF MULTISETS

Let us briefly state several properties of shuffle posets of multisets, most of which follow from the chain decomposition of Section 2. Many are instances of results about $k$-shuffle posets, so we defer those proofs to a later section.

PROPOSITION 3.1. *Shuffle posets of multisets are lattices.*

*Proof.* This is a special case of Theorem 6.1. ∎

PROPOSITION 3.2. *Shuffle posets of multisets are EL-shellable.*

*Proof.* Label the edges with the letters to be inserted or deleted, letting letters to be deleted be smaller than those to be inserted. This is extended to $k$-shuffle posets in Proposition 6.2. ∎

COROLLARY 3.1. *Shuffle posets of multisets are Cohen–Macaulay.*

PROPOSITION 3.3. *If* $\alpha = 1^m$ *and* $\beta = 1^n$, *then* $\mu_{W_{\alpha,\beta}}(\hat{0}, \hat{1}) = (-1)^{m+n} \binom{m+n}{m}$. *Otherwise,* $\mu_{W_{\alpha,\beta}}(\hat{0}, \hat{1}) = 0$.

*Proof.* One may count decreasing chains in the EL-labelling, or alternatively this may be shown using NBB bases, as discussed in Remark 6.1. ∎

Note that every interval is a product of smaller shuffle posets of multisets. Since the Möbius function of a product of posets equals the product of their Möbius functions, $\mu(u, v)$ may be determined from Proposition 3.3 for arbitrary $u \leqslant v$. A canonical way of associating products of shuffle posets to intervals is given in [SS, p. 8]. This extends immediately to shuffle posets of multisets.

When $\mu_P(\hat{0}, \hat{1}) = 0$, shellability implies the collapsibility of the order complex since the reduced homology groups all vanish.

COROLLARY 3.2. *The order complex of* $W_{\alpha,\beta}$ *is collapsible unless* $\alpha = 1^m$ *and* $\beta = 1^n$ *for* $m, n \in \mathbb{N}$.

A **symmetric chain decomposition** of a finite, ranked poset is a decomposition of the poset elements into symmetrically placed saturated chains, by which we mean that the rank of the minimal element of such a saturated chain plus the rank of its maximal element must equal the rank of the poset.

THEOREM 3.1. *Shuffle posets of multisets have symmetric chain decompositions.*

*Proof.* This follows from a restriction of the chain decomposition to 1-chains, as shown in Proposition 6.1. ∎

PROPOSITION 3.4. *Shuffle posets of multisets are supersolvable.*

*Proof.* Any saturated chain including the empty word will turn out to be an $M$-chain, as shown in Theorem 6.3. ∎

Next we provide an analogue to Theorem 3.4 of Greene in [Gr, p. 195]. Let $\Omega_P$ be the number of poset elements, let $\Omega_P(q)$ be the rank generating function, let $C_P$ count maximal chains in $P$, let $Z_P(s)$ be the zeta polynomial, counting multichains $\hat{0} \leqslant x_1 \leqslant \cdots \leqslant x_s = \hat{1}$, let $\chi_P(t)$ be the characteristic polynomial $\sum_{u \in P} \mu(\hat{0}, u)\, t^{n-\mathrm{rk}(u)}$, and let $[n]_q = (1-q^n)/(1-q)$, namely the $q$-analogue of $n$. Let $\chi_{\alpha,\beta}$ denote $\chi_P(t)$ for $P = W_{\alpha,\beta}$.

THEOREM 3.2. *The following formulas hold for the poset $W_{\alpha,\beta}$ in which $\alpha = (\alpha_1, ..., \alpha_k)$, $\beta = (\beta_1, ..., \beta_l)$, $m = \sum_{i=1}^{k} \alpha_i$ and $n = \sum_{i=1}^{l} \beta_i$,*

$$\Omega_{\alpha,\beta} = \sum_{j=0}^{\min(l(\alpha), l(\beta))} \sum_{\substack{1 \leqslant a_1 < \cdots < a_j \leqslant l(\alpha) \\ 1 \leqslant b_1 < \cdots < b_j \leqslant l(\beta)}} \left( \prod_{i=1}^{j} \alpha_{a_i} \beta_{b_i} \right) \times \left( \prod_{i \notin \{a_1, ..., a_j\}} (\alpha_i + 1) \right) \left( \prod_{i \notin \{b_1, ..., b_j\}} (\beta_i + 1) \right)$$

$$\Omega_{\alpha,\beta}(q) = \sum_{j=0}^{\min(l(\alpha), l(\beta))} \sum_{\substack{1 \leqslant a_1 < \cdots < a_j \leqslant l(\alpha) \\ 1 \leqslant b_1 < \cdots < b_j \leqslant l(\beta)}} \times \left( \prod_{i=1}^{j} ([\alpha_{a_i} + 1]_q [\beta_{b_i} + 1]_q - [\alpha_{a_i} + \beta_{b_i} + 1]_q) \right) \times \left( \prod_{i \notin \{a_1, ..., a_j\}} [\alpha_i + 1]_q \right) \left( \prod_{i \notin \{b_1, ..., b_j\}} [\beta_i + 1]_q \right)$$

$$C_{\alpha,\beta} = (m+n)! \sum_{j=0}^{\min(l(\alpha), l(\beta))} \sum_{\substack{1 \leqslant a_1 < \cdots < a_j \leqslant l(\alpha) \\ 1 \leqslant b_1 < \cdots < b_j \leqslant l(\beta)}} \times \left( \prod_{i=1}^{j} \left( \frac{1}{(\alpha_{a_i})!\,(\beta_{b_i})!} - \frac{1}{(\alpha_{a_i} + \beta_{b_i})!} \right) \right) \times \left( \prod_{i \notin \{a_1, ..., a_j\}} \left( \frac{1}{\alpha_i!} \right) \right) \left( \prod_{i \notin \{b_1, ..., b_j\}} \left( \frac{1}{\beta_i!} \right) \right)$$

$$Z_{\alpha,\beta}(s) = \sum_{j=0}^{\min(l(\alpha),\,l(\beta))} \sum_{\substack{1\leqslant a_1<\cdots<a_j\leqslant l(\alpha)\\ 1\leqslant b_1<\cdots<b_j\leqslant l(\beta)}} \times\left(\prod_{i=1}^{j}\binom{\alpha_{s_i}+s-1}{s-1}\binom{\beta_{b_i}+s-1}{s-1}-\binom{\alpha_{a_i}+\beta_{b_i}+s-1}{s-1}\right)$$

$$\times\left(\prod_{i\notin\{a_1,\ldots,a_j\}}\binom{\alpha_i+s-1}{s-1}\right)\left(\prod_{i\notin\{b_1,\ldots,b_j\}}\binom{\beta_i+s-1}{s-1}\right)$$

$$\chi_{\alpha,\beta}(t)=t^{m+n-l(\alpha)-l(\beta)}(t-1)^{l(\alpha)+l(\beta)}\sum_{j=0}^{\min(l(\alpha),\,l(\beta))}\binom{l(\alpha)}{j}\binom{l(\beta)}{j}\left(\frac{1}{1-t}\right)^{j}.$$

*Proof.* This follows from Theorem 2.1 together with Theorem 6.1 of [He], or alternatively from the recurrence relations given in the proof of Theorem 3.2. ∎

THEOREM 3.3. *Summing over compositions $\alpha$ and $\beta$ indexed by monomials in noncommuting variables $u_i$ and $v_j$ with relations $u_i v_j = v_j u_i$ for all $i$, $j>0$ yields*

$$\sum_{\alpha,\beta}\Omega_{\alpha,\beta}u_\alpha v_\beta=\frac{1}{1-\sum_{k>0}(k+1)\,u_k-\sum_{l>0}(l+1)\,v_l+\sum_{k,l>0}(k+l-1)\,u_k v_l}$$

$$\sum_{\alpha,\beta}\Omega_{\alpha,\beta}(q)\,u_\alpha v_\beta =\frac{1}{1-\sum_{k>0}[k+1]_q\,u_k-\sum_{l>0}[l+1]_q\,v_l+\sum_{k,l>0}[k+l-1]_q\,u_k v_l}$$

$$\sum_{\alpha,\beta}\left(\frac{C_{\alpha,\beta}}{(m+n)!}\right)u_\alpha v_\beta=\frac{1}{1-\sum_{k>0}\left(\frac{1}{k!}\right)u_k-\sum_{l>0}\left(\frac{1}{l!}\right)v_l+\sum_{k,l>0}\left(\frac{1}{(k+l)!}\right)u_k v_l}$$

$$\sum_{\alpha,\beta}Z_{\alpha,\beta}(s)\,u_\alpha v_\beta =\frac{1}{1-\sum_{k>0}\binom{s+k-1}{s-1}u_k-\sum_{l>0}\binom{s+l-1}{s-1}u_l+\sum_{k,l>0}\binom{s+k+l-1}{s-1}u_k v_l}$$

$$\sum_{\alpha,\beta}\chi_{\alpha,\beta}(t)\,u_\alpha v_\beta=\frac{1}{1-\sum_{k>0}f(t,k)\,u_k-\sum_{l>0}f(t,l)\,v_l+\sum_{k,l>0}f(t,k+l)\,u_k v_l}$$

*for $f(t,j)=t^{j-1}(t-1)$.*

*Proof.* Let $(\alpha_1, ..., \hat{\alpha}_{a_i}, ..., \alpha_k)$ denote $\alpha - \alpha_{a_i}$ and $(\beta_1, ..., \hat{\beta}_{b_i}, ..., \beta_l)$ denote $\beta - \beta_{b_i}$. The above identities follow from the following recurrences for $\alpha = (\alpha_1, ..., \alpha_k)$ and $\beta = (\beta_1, ..., \beta_l)$.

$$\Omega_{\alpha, \beta} = (\alpha_k + 1)\, \Omega_{\alpha - \alpha_k, \beta} + (\beta_l + 1)\, \Omega_{\alpha, \beta - \beta_l} - (\alpha_k + \beta_l + 1)\, \Omega_{\alpha - \alpha_k, \beta - \beta_l}$$

$$\Omega_{\alpha, \beta}(q) = [\alpha_k + 1]_q\, \Omega_{\alpha - \alpha_k, \beta}(q) + [\beta_l + 1]_q\, \Omega_{\alpha, \beta - \beta_l}(q) - [\alpha_k + \beta_l + 1]_q\, \Omega_{\alpha - \alpha_k, \beta - \beta_l}(q)$$

$$C_{\alpha, \beta} = \binom{m+n}{\alpha_k} C_{\alpha - \alpha_k, \beta} + \binom{m+n}{\beta_l} C_{\alpha, \beta - \beta_l} - \binom{m+n}{\alpha_k + \beta_l} C_{\alpha - \alpha_k, \beta - \beta_l}$$

$$Z_{\alpha, \beta}(s) = \binom{s + \alpha_k - 1}{s - 1} Z_{\alpha - \alpha_k, \beta}(s) + \binom{s + \beta_l - 1}{s - 1} Z_{\alpha, \beta - \beta_l}(s) - \binom{s + \alpha_k + \beta_l - 1}{s - 1} Z_{\alpha - \alpha_k, \beta - \beta_l}(s)$$

$$\chi_{\alpha, \beta}(t) = (t^{\alpha_k - 1}(t-1))\, \chi_{\alpha - \alpha_k, \beta}(t) + (t^{\beta_l - 1}(t-1))\, \chi_{\alpha, \beta - \beta_l}(t) - (t^{\alpha_k + \beta_l - 1}(t-1))\, \chi_{\alpha - \alpha_k, \beta - \beta_l}(t)$$

These recurrences may be proven by induction. ∎

QUESTION 3.1. *Do results of Simion and Stanley [SS, pp. 25–32] about the monoid of multiplicative functions generalize to generating functions in several variables, e.g. the variables $u_i, v_j$ with relations $u_i v_j = v_j u_i$ for all positive integers $i$ and $j$.*

## 4. TWO EQUIVALENT DEFINITIONS FOR *K*-SHUFFLE POSETS

This section introduces shuffle posets for shuffling $k$ words, answering a question of Stanley. The definition will restrict to shuffle posets of multisets which in this context become 2-shuffle posets. The $k$-shuffle posets are defined in such a way that the $i$-shuffle poset $W_{\alpha^{(1)}, ..., \alpha^{(i)}}$ is naturally embedded in the $j$-shuffle poset $W_{\alpha^{(1)}, ..., \alpha^{(j)}}$ for $i < j$.

A $k$-shuffle poset will be specified by $k$ words $w_1, ..., w_k$ to be shuffled. We require that the letters of $w_1, ..., w_k$ come from disjoint alphabets and insist that identical letters within any particular $w_i$ must occur consecutively. A $k$-shuffle poset will be determined up to isomorphism by an ordered set of $k$ compositions specifying the types of the words $w_1, ..., w_k$ to be shuffled. Letting $\alpha^{(i)}$ be the type of the word $w_i$ for $1 \leqslant i \leqslant k$, then $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ denotes the $k$-shuffle poset in which $w_1, ..., w_k$ are shuffled.

One might expect each $k$-shuffle poset element to come from shuffling the words $w_1, ..., w_k$ and then choosing a subword. However, it is not clear how to partially order such shuffled subwords in a way that will yield a poset. Therefore, we define each $k$-shuffle poset element to be a $(k+1)$-tuple consisting of a subword $u_i$ of the word $w_i$ for $1 \leqslant i \leqslant k$ together with a collection of pairwise shuffled words. For each $1 \leqslant i < j \leqslant k$, we specify how to shuffle the complement of $u_i$ (viewed as a subword of $w_i$) with $u_j$; the resulting shuffled words comprise this collection of pairwise shuffled words. To give rise to a poset element, we require such a collection of pairwise shuffled words to be consistent, as defined next.

Let $u_i^c$ denote the complement of $u_i$ within $w_i$ and let $u_i \sqcup\!\sqcup\, u_j$ be a word obtained by shuffling $u_i$ and $u_j$.

Definition 4.1. A collection $\{u_i^c \sqcup\!\sqcup\, u_j \mid i < j\}$ of pairwise shuffled words is **consistent** if there is some shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup\, w_k$ which contains each $u_i^c \sqcup\!\sqcup\, u_j$ as a subword.

Each covering relation amounts to inserting a letter with respect to all "earlier" words and at the same time deleting it with respect to all "later" words, by way of an operation which we therefore call del-sertion; we require the union of the original collection of pairwise shuffled words and the new collection of pairwise shuffle words to all be consistent for a covering relation to result.

Definition 4.2. Let $C$ be a consistent collection $\{u_i^c \sqcup\!\sqcup\, u_j \mid i < j\}$ of pairwise shuffled words and let $b$ be a letter belonging to some $u_l$. Then $b$ is **del-serted** by deleting $b$ from each copy of $u_l^c$ and at the same time inserting $b$ in each copy of $u_l$ in $C$.

In summary, we have the following definition.

Definition 4.3. The **k-shuffle poset** $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ is given by the following elements and covering relations.

(1) Let $u$ be an element of $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ if $u = (u_1, ..., u_k, \{u_i^c \sqcup\!\sqcup\, u_j \mid i < j\})$, where $u_i$ is a subword of $w_i$ for $1 \leqslant i \leqslant k$ and the collection of pairwise shuffled words $\{u_i^c \sqcup\!\sqcup\, u_j \mid i < j\}$ is consistent.

(2) Let $u \prec v$ if $v$ is obtained from $u$ by del-serting a letter from some $u_i$ in such a way that $\{u_i^c \sqcup\!\sqcup\, u_j \mid i < j\} \cup \{v_i^c \sqcup\!\sqcup\, v_j \mid i < j\}$ is consistent.

The consistency requirement on covering relations $u \prec v$ is automatic for $k = 2$, but a necessary assumption for larger $k$. For example, let $w_1 = 1$, $w_2 = b$, $w_3 = C$ and let $\varnothing$ denote the empty word. If $u = (\varnothing, \varnothing, C, \{1, 1C, Cb\}$ and $v = (\varnothing, b, C, \{b1, 1C, C\}$, then one might hope to obtain $v$ from

$u$ by del-serting $b$, but this is not allowed because $u$ and $v$ are not consistent. See Fig. 4 for the entire 3-shuffle poset in this case. We label poset elements with the sets of shuffled words $\{u_1^c \sqcup\!\sqcup u_2, u_1^c \sqcup\!\sqcup u_3, u_2^c \sqcup\!\sqcup u_3\}$. Proposition 4.1 will show that edge consistency implies consistency of all poset chains.

Let us verify (through a somewhat technical lemma) that each poset chain is consistent with at least one shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$.

DEFINITION 4.4. A chain contains a **loop** $a_1, ..., a_m$ if the letters $a_1, ..., a_m$ all occur in pairwise shuffled words in the chain in such a way that each $a_i$ precedes $a_{i+1}$ in some pairwise shuffled word in the chain, and if $a_m$ also precedes $a_1$ in some chain element.

If there is an inconsistency in a saturated chain, there must be a loop. If a letter $a_i$ is del-serted in a covering relation $u \prec v$, then let $t(a_i)$ be the rank of $v$, since in some sense this as the time at which $a_i$ is del-serted in travelling from $\hat{0}$ to $\hat{1}$. Let $w(a_i)$ be the index of the word to which $a_i$ belongs, namely if $a_i \in w_j$ for $1 \leqslant j \leqslant k$ then $w(a_i) = j$. We say that one letter precedes another at $t(a_i)$ if this is true in any of the pairwise shuffled words in either $u$ or $v$.

PROPOSITION 4.1. *Every chain in a k-shuffle poset is consistent with at least one way of shuffling* $w_1, ..., w_k$.

*Proof.* It suffices to verify this for saturated chains. This will amount to showing that whenever a saturated chain has an inconsistency, there is

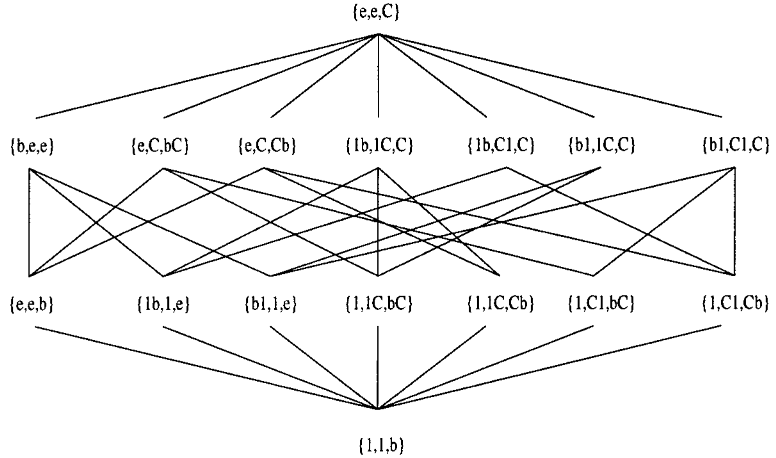


**FIG. 4.** An example of a $k$-shuffle poset.

some inconsistent edge in the chain. To simplify notation, we assume that each letter occurs with multiplicity one because the general case is essentially the same.

Suppose the letters $a_1, ..., a_m$ form a loop, but every edge is consistent. Without loss of generality, we may assume $t(a_m) > t(a_1)$. This implies $w(a_m) \geqslant w(a_1)$ since $a_1$ and $a_m$ have comparable positions at some time. Similarly, note that $t(a_i) < t(a_{i+1})$ implies $w(a_i) \leqslant w(a_{i+1})$ and $t(a_i) > t(a_{i+1})$ implies $w(a_i) \geqslant w(a_{i+1})$ for $1 \leqslant i \leqslant m-1$. If $t(a_i) > t(a_m)$ and $w(a_i) \geqslant w(a_m)$ for some $1 < i < m$, then we get a loop $a_1, a_i, a_m$ at time $t(a_i)$. Similarly, we cannot have $t(a_i) < t(a_1)$ and $w(a_i) \leqslant w(a_1)$ for $1 < i < m$. In particular, $t(a_2) > t(a_1)$ and $t(a_{m-1}) < t(a_m)$.

Let us first consider the case $t(a_m) = t(a_1) + 1$. This implies $t(a_2) > t(a_m)$ and $w(a_1) \leqslant w(a_2) < w(a_m)$. By the same reasoning, $t(a_{m-1}) < t(a_1)$ and $w(a_1) < w(a_{m-1}) \leqslant w(a_m)$. The Intermediate Value Theorem then implies the existence of some $1 < i < m$ such that $t(a_i) > t(a_m) > t(a_1) > t(a_{i+1})$. We must have $w(a_1) \leqslant w(a_{i+1}) \leqslant w(a_i) \leqslant w(a_m)$; otherwise we would have $w(a_{i+1}) < w(a_1)$ or $w(a_i) > w(a_m)$, either of which would lead to a loop of size three. However, the inequalities $w(a_1) \leqslant w(a_{i+1}) \leqslant w(a_i) \leqslant w(a_m)$ imply a loop $a_1, a_i, a_{i+1}, a_m$, giving an inconsistent edge at $t(a_1)$.

When $t(a_m) > t(a_1) + 1$, the same argument applies unless there is some $1 < i < m$ such that $t(a_1) < t(a_i) < t(a_m)$. If so, $w(a_i) > w(a_m)$ or $w(a_i) < w(a_1)$. Without loss of generality, assume the former. Since $w(a_{m-1}) < w(a_m)$, there exists some $j \geqslant i$ such that $w(a_j) > w(a_m) > w(a_{j+1})$, while $t(a_j) > t(a_1)$ and $t(a_{j+1}) < t(a_m)$. This gives rise to a loop $a_1, a_j, a_{j+1}, a_m$ which yields an inconsistency at $t(a_j)$. Hence, there must always be an inconsistent edge. ∎

Next we give a more complicated constructive definition for $k$-shuffle posets and check its equivalence to Definition 4.3. First note that each possible shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to a product of chains subposet of the form $C_\alpha$ where $\alpha$ is the composition obtained by taking the union of the compositions $\alpha^{(1)}, ..., \alpha^{(k)}$ for $w_i$ of type $\alpha^{(i)}$. For each $w$, we denote by $P_w$ the product of chains subposet of those elements of $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ which are consistent with $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$.

Each shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ may equivalently be represented by a consistent collection of pairwise shuffled words $\{w_i \sqcup\!\sqcup w_j \mid i < j\}$. If we let $e_j$ denote the empty word considered as a subword of $w_j$, then each product of chains will have a collection $\{w_i \sqcup\!\sqcup e_j \mid i < j\}$ as its minimal element. Each covering relation will amount to del-serting a letter in the unique way that is consistent with the shuffled word specifying the product of chains. Thus, the labels on covering relations in Definition 4.5 may be viewed as the letters to be del-serted.

DEFINITION 4.5. The $k$-shuffle poset $W_{\alpha^{(1)}, \ldots, \alpha^{(k)}}$ is constructed as follows.

(1) Let $\hat{0} = \{w_i \sqcup\!\sqcup e_j \mid i < j\}$.

(2) If $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ is a shuffling of $w_1, \ldots, w_k$, then label covering relations within the product of chains $P_w$ by letters in $w_1, \ldots, w_k$ in the natural way. Namely, if $u \prec v$ and $v$ differs from $u$ in the coordinate in position $l(\alpha^{(1)}) + \cdots + l(\alpha^{(i)}) + j$ in $P_w$, then the covering relation $u \prec v$ is labelled with the $j$th distinct letter in $w_{i+1}$. Label $v$ with the collection of pairwise shuffled words obtained by del-serting the label of the edge $(u, v)$ into the collection of pairwise shuffled words for $u$.

(3) Glue together two elements from distinct products of chains $P_w$ and $P_{w'}$ if they are specified by identical collections of pairwise shuffled words. Glue together two covering relations $u \prec v$ and $u' \prec v'$ if $u$ is glued to $u'$ and $v$ is glued to $v'$.

PROPOSITION 4.2. *Definitions* 4.3 *and* 4.5 *are equivalent.*

*Proof.* In Definition 4.5, saturated chains are given by the order in which labels occur and by the collection of shuffled words $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ with which they are consistent; Proposition 4.5 will ensure that every saturated chain is consistent with at least one shuffled word. The label order in a chain given by Definition 4.3 determines del-sertion order in a Definition 4.3 chain, and the positions in which letters are del-serted is completely determined by the collection of shuffled words with which the chain is consistent. Identification of elements from distinct products of chains in Definition 4.5 is tantamount to deletion before insertion in Definition 4.3. This map of chains induces an order-preserving bijection between poset elements. ∎

## 5. A FLAG $f$-VECTOR FORMULA FOR $k$-SHUFFLE POSETS

The chain decomposition for $k$-shuffle posets is quite similar to that of shuffle posets of multisets, but interface pairs are replaced by what we call descent blocks and in $k$-shuffle posets the ribbon shapes may involve as many as $k$ rows. Recall how each shuffled word $w = w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to a product of chains sublattice $P_w$. We will again partially order these using an interval in the weak order. Let $l(w_i)$ be the **length** of the composition $\alpha^{(i)}$ which records the type of the word $w_i$. We use the interval in the weak order from the permutation $(l(w_1), \ldots, 1,\ l(w_1) + l(w_2), \ldots, l(w_1) + 1, \ldots, l(w_1) + \cdots + l(w_k), \ldots, l(w_1) + \cdots + l(w_{k-1}) + 1)$ to the reverse permutation $(l(w_1) + \cdots + l(w_k), \ldots, 1)$; as before, we have covering relations from swapping adjacent positions rather than values. The point is to

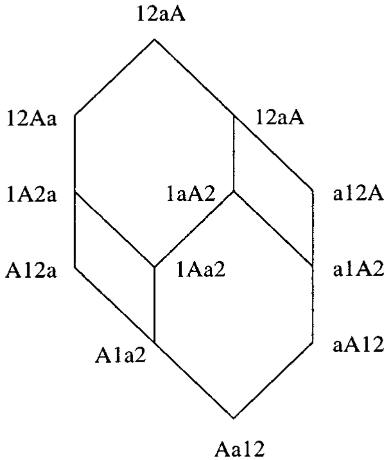


**FIG. 5.** The partial order on sublattices indexed by shuffled words.

preserve the order of the letters belonging to each word $w_i$. For example, if $w_1 = 112$, $w_2 = aaa$ and $w_3 = A$, the weak order interval we use is given in Fig. 5.

Mimicking the case $k = 2$, each poset chain is assigned to $P_w$ for the earliest $w$ which is consistent with the chain. Proposition 4.1 ensures that every chain belongs to some $P_w$. We also need to make sure that this choice is well-defined of which $P_w$ containing a chain is earliest. This follows from a generalized notion of the interface of a chain which we will call the set of descent blocks of the chain.

DEFINITION 5.1. A **descent block** is a maximal string $u_1 \cdots u_j$ of consecutive letters (ignoring repetition of identical letters) in a shuffled word with the property that $w(u_i) > w(u_{i+1})$ for $1 \leqslant i < j$.

For example, if we replace a word $u = u_1 \cdots u_n$ by $w(u_1) \cdots w(u_n)$ to obtain 3114214241, then $u$ has descent blocks represented by 31, 1, 421, 42, and 41. Let $m_i(\mathbf{b})$ be the multiplicity with which the $i$th distinct letter in a descent block $\mathbf{b}$ occurs, let $l(\mathbf{b})$ be the number of distinct letters in $\mathbf{b}$ and let $S_{(m_1(\mathbf{b}), \ldots, m_j(\mathbf{b}))}$ denote the skew-Schur function of ribbon shape with $m_i(\mathbf{b})$ boxes in row $l(\mathbf{b}) - i + 1$. We claim that the collection of multichains assigned to a product of chains with $B$ as its set of descent blocks contributes

$$\prod_{\mathbf{b} \in B} S_{(m_1(\mathbf{b}), \ldots, m_{l(b)}(\mathbf{b}))}$$

to $F_P$. For example,

$$S_{(3,1,2)} = s_{\ydiagram{2+2,2+1,3}}$$

would account for a descent block $CBA$ in which $w(C) > w(B) > w(A)$, $m_1(\mathbf{b}) = 3$, $m_2(\mathbf{b}) = 1$ and $m_3(\mathbf{b}) = 2$.

If $B$ is the set of descent blocks for some $P_w$ then the multichains within $P_w$ which are assigned to it are the multichains which actually involve all the descents in the descent blocks of $B$. This is exactly the collection of multichains in $P_w$ determined by the system of weak and strong inequalities specifying a product of skew-Schur functions of ribbon shape, each of which has $l(\mathbf{b})$ rows, so Lemma 1.1 applies. A descent block consisting of letters $a_1, ..., a_l$ with multiplicities $m_1, ..., m_l$ will contribute to $F_P$ the skew-Schur function $S_{(m_1, ..., m_l)}$, and the contribution of the descent blocks for some $P_w$ are multiplied. Simply note that we have weak inequalities on the order in which identical letters are del-serted and strict inequalities requiring that the last copy of $a_i$ must be del-serted strictly after the first copy of $a_{i+1}$ is del-serted. Otherwise, not all of the necessary descents would occur in the multichain.

EXAMPLE 5.1. If $w_1 = aabbbcddddde$, $w_2 = ABBCDDD$, $w_3 = 1112333344$, then the chains associated to shuffled word

$$aa111bbbcA2BBdddddCe333344DDD$$

contribute the product

$$s_{\ydiagram{2}}\, s_{\ydiagram{2+3,3}}\, s_{\ydiagram{1}}\, s_{\ydiagram{1}}\, s_{\ydiagram{1+5,2,1}}\, s_{\ydiagram{1,1}}\, s_{\ydiagram{4}}\, s_{\ydiagram{1+3,2}}$$

of skew-Schur functions to $F_P$. The descent blocks are $a$, $1b$, $c$, $A$, $2Bd$, $Ce$, $3$ and $4D$.

THEOREM 5.1. *Let* $\mathrm{Shuf}(w_1, ..., w_k)$ *be the collection of shuffled words obtained by shuffling* $w_1, ..., w_k$ *and let* $B(w)$ *be the collection of descent blocks in a particular* $w \in \mathrm{Shuf}(w_1, ..., w_k)$. *Then*

$$F_P = \sum_{w \in \mathrm{Shuf}(w_1, ..., w_k)} \prod_{\mathbf{b} \in B(w)} S_{(m_1(\mathbf{b}), ..., m_{l(b)}(\mathbf{b}))}.$$

*Proof.* This is an immediate consequence of Lemma 1.1, in light of the above discussion. ∎

COROLLARY 5.1. *The flag f-vector formula $F_P$ for k-shuffle posets is Schur-positive.*

*Proof.* The proof is similar to the $k=2$ case, since we have again expressed $F_P$ as a sum of products of skew-Schur functions. ∎

To express $F_P$ in terms of complete symmetric functions, note that a term $h_{m_1} \cdots h_{m_j}$ comes from filling each row in a skew-tableau of ribbon shape with a weakly increasing sequence, but terms must be subtracted to account for the fact that a box vertically above another cannot have a (weakly) larger entry, in which case the two rows concatenated would form a weakly increasing sequence; inclusion-exclusion thus leads to an alternating sum of complete homogeneous symmetric functions for each ribbon shape. Complete symmetric functions are useful in giving recurrences for $F_P$.

Recall our notation $\bar{\alpha}$ for the composition $(\alpha_1, ..., \alpha_{k-1})$ obtained from $\alpha = (\alpha_1, ..., \alpha_k)$ by deleting the last part. Let $F(S, \alpha^1, ..., \alpha^k)$ be $F_P$ for the $k$-shuffle poset obtained from $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ by replacing $\alpha^{(j)}$ by $\overline{\alpha^{(j)}}$ for each $j \in S$.

PROPOSITION 5.1. *The k-shuffle posets satisfy the recurrence*

$$F_{\alpha^{(1)}, ..., \alpha^{(k)}} = \sum_{S \subseteq \{1, ..., k\}} (-1)^{|S|-1} F(S, \alpha^{(1)}, ..., \alpha^{(k)}).$$

*Proof.* Recall that a chain is $w_i$-terminal if the last letter of $w_i$ always occurs last in pairwise shuffled words in the chain that contain the letter. Proposition 4.5 shows that each chain is consistent with at least one shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$; in particular, this means that the chain is $w_i$-terminal for some nonempty collection of indices $i$ which we call $S$. Such a chain will contribute to each summand on the right side which is indexed by any subset $T \subseteq S$. The coefficients for these summands are the Möbius functions of the boolean lattice of subsets of $S$, each multiplied by $-1$. Note that the empty set is the only subset of $S$ not occurring, and $1 = \sum_{\varnothing \subset T \subseteq S} -\mu(\hat{0}, T)$, so each chain is accounted for exactly once. ∎

Let $u_\alpha$ denote the word $u_{\alpha_1} \cdots u_{\alpha_k}$ in the following proposition.

PROPOSITION 5.2. *The sum $\sum_{\alpha, \beta, \gamma} F_{\alpha, \beta, \gamma} u_\alpha v_\beta w_\gamma$ over all possible 3-tuples of compositions equals*

$$\Bigg(1 - \sum_{i>0} (u_i + v_i + w_i)\, h_i + \sum_{i, j>0} (u_i v_j + u_i w_j + v_i w_j)\, h_{i+j} - \sum_{i, j, k>0} u_i v_j w_k h_{i+j+k}\Bigg)^{-1}.$$

*More generally, summing over all k-tuples of compositions yields*

$$\sum_{\alpha^{(1)}, ..., \alpha^{(k)}} F_{\alpha^{(1)}, ..., \alpha^{(k)}} u^{(1)}_{\alpha(1)} \cdots u^{(k)}_{\alpha(k)}$$
$$= \left(1 - \sum_{i=1}^{k} \sum_{j_1, ..., j_i > 0} h_{j_1 + \cdots + j_i} \sum_{1 \leqslant t_1 < \cdots < t_i \leqslant k} u^{(t_1)}_{j_1} \cdots u^{(t_i)}_{j_i}\right)^{-1},$$

*where the expressions $F_{\alpha^{(1)}, ..., \alpha^{(k)}}$ are indexed by monomials in the alphabets $u^{(1)}, ..., u^{(k)}$ in noncommuting variables $u^{(i)}_j$ satisfying relations $u^{(i_1)}_{j_1} u^{(i_2)}_{j_2} = u^{(i_2)}_{j_2} u^{(i_1)}_{j_1}$. The monomial $u^{(i)}_{\alpha^{(i)}}$ is shorthand for $u^{(i)}_{\alpha^{(i)}_1} \cdots u^{(i)}_{\alpha^{(i)}_l}$, where $l = l(\alpha^{(i)})$.*

*Proof.* This follows from the recursive formula given in Proposition 5.1 just as in the case $k = 2$. ∎

The definition of $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ depends on the order in which the compositions $\alpha^{(1)}, ..., \alpha^{(k)}$ are arranged, so Greene asked if the rank generating function also depends on the order of the $k$ words to be shuffled [Gr2]. The following implies that it does not, and furthermore that the flag $f$-vector does not.

COROLLARY 5.2. *If $w_i$ is of type $\alpha^{(i)}$ for $1 \leqslant i \leqslant i$, then*

$$F_{\alpha^{(1)}, \alpha^{(2)}, ..., \alpha^{(k)}} = F_{\alpha^{\sigma(1)}, ..., \alpha^{\sigma(k)}}$$

*for any $\sigma \in S_k$ permuting the k compositions specifying the types of the words to be shuffled.*

*Proof.* This is an immediate consequence of the symmetry of the expression for $\sum_{\alpha, \beta, \gamma} F_{\alpha, \beta, \gamma} u_\alpha v_\beta w_\gamma$ given in Proposition 5.2 in the alphabets $u^{(1)}, ..., u^{(k)}$. ∎

## 6. PROPERTIES OF $k$-SHUFFLE POSETS

THEOREM 6.1. *The k-shuffle poset $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ is a lattice.*

*Proof.* Let $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ be the $k$-shuffle poset given by words $w_1, ..., w_k$ where $w_i$ is of type $\alpha^{(i)}$. We will construct $u \vee v$ in such a way that its minimality will be clear. Let us describe how to del-sert letters in $u$ so as to obtain the smallest possible poset element which is also greater than $v$. We will specify which letters occur in which positions in each pairwise shuffled word assuming each letter in $w_1, ..., w_k$ occurs with multiplicity one; the reader may consult our proof for shuffle posets of multisets to find a convention to handle multiplicity that generalizes directly to $k$-shuffle posets.

We proceed by induction, describing how to del-sert letters belonging to $w_i$ from $u$ assuming that we are done del-serting letters belonging to $w_1, ..., w_{i-1}$ and have not yet del-serted any letters belonging to $w_j$ for $j > i$. Any letter of $w_1$ that has been del-serted in $v$ but not in $u$ should be del-serted from $u$. If $u_1^c \sqcup\sqcup u_i$ is inconsistent with $v_1^c \sqcup\sqcup v_i$ for some $i > 1$, then there exists $a \in w_1$ and $b \in w_i$ such that $a$ precedes $b$ in $u_1^c \sqcup\sqcup u_i$ while $b$ precedes $a$ in $v_1^c \sqcup\sqcup v_i$, or vice-versa. In either case, we del-sert $a$ from $u$. These are the only letters of $w_1$ to be del-serted from $u$.

Assume that we have del-serted letters belonging to $w_1, ..., w_{j-1}$ as necessary from $u$ to obtain $\dot{u} \geqslant u$. We next del-sert from $\dot{u}$ any letter $b$ of $w_j$ which has been del-serted from $v$. We will call the result $\ddot{u}$. To do this, we need to specify where to insert $b$ into $\dot{u}_i^c \sqcup\sqcup \dot{u}_j$ for each $i < j$. However, there is a unique way to do this which is consistent with $v_i^c \sqcup\sqcup v_j$, because $\dot{u}_i^c$ is a subword of $v_i^c$.

If $\ddot{u}_j^c \sqcup\sqcup u_k$ is inconsistent with $v_j^c \sqcup\sqcup v_k$ for some $k > j$, then again there exists $a \in w_j$ and $b \in w_k$ such that $a$ precedes $b$ in $\ddot{u}_j^c \sqcup\sqcup u_k$ while $b$ precedes $a$ in $v_j^c \sqcup\sqcup v_k$, or vice-versa. All such letters $a$ belonging to $w_j$ need to be del-serted from $\ddot{u}$. There is a unique position in which to insert $a$ into $\ddot{u}_i^c \sqcup\sqcup u_j$ for $i < j$. This is based on the position of $b$ in $\ddot{u}_i^c \sqcup\sqcup u_k$; namely, $a$ must be inserted between the same two letters of $\ddot{u}_i^c$ that $b$ is between in $\ddot{u}_i^c \sqcup\sqcup u_k$. This is because $u \vee v$ must be consistent with a pairwise shuffled word in $u$ which has $a$ preceding $b$ and also with a pairwise shuffled word in $v$ which has $b$ preceding $a$, or vice-versa, so $a$ and $b$ must occur in incomparable positions in $u \vee v$. By this algorithm, we construct $u \vee v$. ∎

THEOREM 6.2. *The k-shuffle posets are EL-shellable.*

*Proof.* Following Proposition 3.2, we label each edge with the letter to be del-serted. The labels need only be ordered in such a way that $a < b$ whenever $w(a) < w(b)$. ∎

This immediately implies the following.

COROLLARY 6.1. *The k-shuffle posets are Cohen–Macaulay.*

COROLLARY 6.2. *If $w_i$ is of type $1^{m_i}$ for $1 \leqslant i \leqslant k$, then the k-shuffle poset given by words $w_1, ..., w_k$ satisfies*

$$\mu_P(\hat{0}, \hat{1}) = (-1)^{\mathrm{rk} P} \binom{m_1 + \cdots + m_k}{m_1, m_2, ..., m_k}.$$

*For all other k-shuffle posets, $\mu_P(\hat{0}, \hat{1}) = 0$.*

*Proof.* Let $n$ be the rank of $P$. Whenever $F_P$ is a symmetric function, the Möbius function $\mu_P(\hat{0}, \hat{1})$ is the coefficient of $h_{1^n}$ in $F_P$ [He], so the result follows from an examination of this coefficient. Alternatively, we may easily count the decreasing chains in our EL-labelling. These are in bijection with the distinct ways of shuffling $w_1, ..., w_k$ if each letter has multiplicity 1, and there are no decreasing chains otherwise. ∎

COROLLARY 6.3. *The k-shuffle posets have collapsible order complex, except when each letter occurs with multiplicity one. In this case, the distinct ways of shuffling the k-words index the cycles in a homology basis.*

*Proof.* The Möbius function is the alternating sum of the ranks of the reduced homology groups, but these vanish except possibly in top dimension since the order complex is Cohen–Macaulay. Hence, the reduced homology groups all vanish when the Möbius function is 0. When these groups all vanish and the complex is shellable, this implies collapsibility.

When each letter occurs with multiplicity one, then each way of shuffling the $k$ words gives rise to a boolean sublattice, which in turn contributes a cycle to the top homology of the order complex. These cycles are indexed by the decreasing chains in the EL-labeling given in Proposition 6.2. ∎

Let $A(L)$ be the set of atoms (i.e., elements of rank 1) in a lattice $L$, and let $\trianglelefteq$ be a partial order on $A(L)$. A set $D \subseteq A(L)$ is bounded below if for all $d \in D$ there exists $a \in A(L)$ such that $a \triangleleft d$ and $a < \bigvee D$. A set $B$ of atoms is NBB if $B$ does not contain any $D$ which is bounded below, and an NBB set satisfying $\hat{1} = \bigvee B$ is an NBB basis.

*Remark* 6.1. An NBB basis is given for the traditional shuffle posets in [BlS, p. 106]. This generalizes in a natural way to $k$-shuffle posets in which each letter occurs with multiplicity one. The partial order on atoms is based on the letter to be del-serted. If an atom $a_1$ del-serts a letter of $w_i$ while another atom $a_2$ del-serts a letter of $w_j$ for $i<j$, then $a_1 \trianglelefteq a_2$, and otherwise the two atoms are incomparable. Each shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to an NBB set consisting of all atoms which are consistent with this shuffled word; these give the whole basis of NBB sets.

Furthermore, the $k$-shuffle posets are supersolvable.

THEOREM 6.3. *The k-shuffle posets are supersolvable.*

*Proof.* Consider any chain $C$ in which all the letters of $w_i$ are del-serted before any of the letters of $w_j$, for each $i<j$. We will show that $C$ must then be an $M$-chain. Note that $W_{\alpha^{(1)}, ..., \alpha^{(k)}}$ may be decomposed into overlapping products of chains $P_w$, all of which include $C$, since $C$ is consistent with

every possible shuffled word. This means each poset chain belongs to some common $P_w$ with $C$, so then the claim follows from the modularity of each $P_w$. ∎

PROPOSITION 6.1. *The k-shuffle posets have symmetric chain decompositions.*

*Proof.* Restricting the chain decomposition of Section 5 to poset elements considered as 1-chains yields a decomposition into symmetrically placed products of chains. Each shuffled word $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ gives rise to a collection of descent blocks $B$, as discussed in the computation of $F_P$. Let $a_1, ..., a_j$ be the distinct letters in a descent block, ordered so that $w(a_i) < w(a_{i'})$ for $i < i'$. Let $m(a_i)$ be the multiplicity with which the letter $a_i$ occurs. We index identical copies of $a_i$ by $a_{i_1}, ..., a_{i_{m(a_i)}}$. Poset elements to be assigned to the piece of the decomposition specified by $w_1 \sqcup\!\sqcup \cdots \sqcup\!\sqcup w_k$ are those elements satisfying the constraints $a_{i_1} < a_{i-1_{m(a_{i-1})}}$ for $1 < i \leqslant j$ for each descent block in the word specifying this piece of the decomposition. These poset elements again form a symmetrically placed product of chains, because of symmetry in the constraints and because we are only considering poset elements in some product of chains $P_w$. ∎

One may obtain the rank generating function, characteristic polynomial and zeta polynomial for $k$-shuffle posets by expressing $F_P$ in terms of complete symmetric functions (by way of the combinatorial definition of skew-Schur function of ribbon shape, as in the proof of Theorem 5.1). Theorem 6.1 of [He] makes this relationship explicit. However, the resulting formulas for $k$-shuffle posets would be sufficiently unwieldy that we do not include them.

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